THE

MATHEMATICAL GAZETTE.

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LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

VOL. II. MARCH, 1902. No. 32.

THE TRIGONOMETRY OF THE TETRAHEDRON.

So far as I know the properties of a tetrahedron are not given in any mathematical text-book. In the late Dr. Wolstenholme's *Examples for Practice in the use of Seven-figure Logarithms*, the formulae which are necessary to determine the plane and dihedral angles and the volume of the tetrahedron, having given the six edges, are stated, but nothing beyond this. Dr. Wolstenholme worked at the subject for some years, and several of his results were sent as problems to the *Educational Times*, where they appeared for the most part after his death. In one set of questions he remarks that they are "the equations I have been looking for for years," and so I think we may fairly infer that these questions represent the more advanced results of his investigations. The solutions of these questions, so far as I have been able to find them, are practically all given in this paper. With regard to the rest of the matter, I hardly dare claim any originality. Wolstenholme must have known all or nearly all the results, and any one seriously attacking the subject must come upon them. All I can say is that I have never seen them in print, and that under any circumstances the bringing of them together must be helpful to students in many branches of mathematics.

The following notation will be used:

$OABC$ is the tetrahedron; V its volume.
$\Delta_0, \Delta_1, \Delta_2, \Delta_3$ the areas of the faces opposite to O, A, B, C respectively.
a, b, c, x, y, z the lengths of OA, OB, OC, BC, CA, AB.
p_0, p_1, p_2, p_3 the perpendiculars from O, A, B, C on the opposite faces.
d_1, d_2, d_3 the joins of the mid-points of a and x, b and y, c and z.
l, m, n the perpendicular distances between a and x, b and y, c and z.
$\alpha_1, \beta_2, \gamma_3$ the angles BOC, COA, AOB;
$\alpha_0, \beta_3, \gamma_2$ " BAC, OAB, CAO;
$\alpha_3, \beta_0, \gamma_1$ " OBA, ABC, CBO;
$\alpha_2, \beta_1, \gamma_0$ " OCA, BCO, ACB.

It should be noticed that

α is *not* adjacent to either a or x,
β " " b or y,
γ " " c or z,

and that the subscripts are the same as those of the faces Δ in which the angles lie.

A, B, C, X, Y, Z the dihedral angles whose edges are a, b, c, x, y, z.
λ, μ, ν, and their supplements, the angles between a and x, b and y, c and z.
θ_1, ϕ_2, ψ_3 the inclinations of a and Δ_1, b and Δ_2, c and Δ_3;
θ_0, ϕ_3, ψ_2 " " a and Δ_0, y and Δ_3, z and Δ_2;
θ_3, ϕ_0, ψ_1 " " x and Δ_3, b and Δ_0, z and Δ_1;
θ_2, ϕ_1, ψ_0 " " x and Δ_2, y and Δ_1, c and Δ_0.
Again it should be noticed that the angles
θ are the inclinations of a and x to the faces opposite to them;
ϕ " " " b and y " " "
ψ " " " c and z " " "
and that the subscripts are the same as those of the faces Δ, to which the edges are respectively inclined.
The fraction $4\Delta_0\Delta_1\Delta_2\Delta_3/9V^2$ will be called h^2.
R, r, r_0, r_1, r_2, r_3 the radii of the circum-sphere, the in-sphere, and the four ex-spheres opposite to O, A, B, C respectively.

1. Speaking generally the tetrahedron is completely determined when we know six parts, one of which must be a length or an area; and so five angles are required in order to determine the other angles. We have however the four necessary relations

$$\alpha_0+\beta_0+\gamma_0=\alpha_1+\beta_1+\gamma_1=\alpha_2+\beta_2+\gamma_2=\alpha_3+\beta_3+\gamma_3=\pi. \quad \ldots\ldots\ldots\ldots(1)$$

$\alpha_1, \beta_2, \gamma_3$ are the sides; A, B, C the opposite angles; and θ_1, ϕ_2, ψ_3 the perpendiculars from the angles on the opposite sides of a spherical triangle, which we shall speak of as the spherical triangle at O.
The corresponding parts of the spherical triangle

at A are $\alpha_0, \beta_3, \gamma_2, A, Y, Z, \theta_0, \phi_3, \psi_2$;
at B are $\alpha_3, \beta_0, \gamma_1, X, B, Z, \theta_3, \phi_0, \psi_1$;
at C are $\alpha_2, \beta_1, \gamma_0, X, Y, C, \theta_2, \phi_1, \psi_0$.

The identities

$$1-\Sigma\cos^2\alpha+2\Pi\cos\alpha=\begin{vmatrix} 1, & \cos\beta, & \cos\gamma \\ \cos\beta, & 1, & \cos\alpha \\ \cos\gamma, & \cos\alpha, & 1 \end{vmatrix}$$

and

$$1-\Sigma\cos^2 A-2\Pi\cos A=-\begin{vmatrix} -1, & \cos B, & \cos C \\ \cos B, & -1, & \cos A \\ \cos C, & \cos A, & -1 \end{vmatrix}$$

will be found useful.

2. We commence by giving the relations between the different angles at a vertex. Neglecting the subscripts and calling the dihedral angles A, B, C, as if the vertex was O, we have from the ordinary formula of spherical trigonometry,

$$\cos A=(\cos\alpha-\cos\beta\cos\gamma)/(\sin\beta\sin\gamma), \text{ etc.}, \quad \ldots\ldots\ldots\ldots(2)$$

$$\sin A/\sin\alpha=\ldots=\ldots=\sqrt{1-\Sigma\cos^2\alpha+2\Pi\cos\alpha}/\Pi\sin\alpha$$
$$=2\sqrt{\sin\sigma\Pi\sin(\sigma-\alpha)}/\Pi\sin\alpha, \quad \ldots\ldots\ldots\ldots(3)$$

$$\tan\frac{A}{2}\sin(\sigma-\alpha)=\ldots=\ldots=\sqrt{\Pi\sin(\sigma-\alpha)/\sin\sigma}, \quad \ldots\ldots\ldots\ldots(4)$$

where $2\sigma=\Sigma\alpha$.
The last set are most useful for numerical calculations.

3. Similarly the plane angles are given in terms of the dihedral angles,

$$\cos\alpha=(\cos A+\cos B\cos C)/\sin B\sin C, \text{ etc.}, \quad \ldots\ldots\ldots\ldots(5)$$

$$\sin\alpha/\sin A=\ldots=\ldots=\sqrt{1-\Sigma\cos^2 A-2\Pi\cos A}/\Pi\sin A, \quad \ldots\ldots\ldots\ldots(6)$$

with corresponding equations for $\tan\frac{\alpha}{2}$, etc.

4. From the spherical triangle we have the equations

$$\sin\theta = \sin\beta \sin C = \sin\gamma \sin B,$$

with similar equations for $\sin\phi$ and $\sin\psi$.

From (3) and (6),

$$\sin\theta \sin\alpha = \sin\phi \sin\beta = \sin\psi \sin\gamma = \sqrt{1 - \Sigma\cos^2\alpha + 2\Pi\cos\alpha}, \quad \ldots\ldots(7)$$

$$\sin\theta \sin A = \sin\phi \sin B = \sin\psi \sin C = \sqrt{1 - \Sigma\cos^2 A - 2\Pi\cos A}. \quad \ldots\ldots(8)$$

Hence, given the three plane or dihedral angles at any vertex, the remaining angles are easily determined.

5. If the three angles θ, ϕ, ψ are given, the determination of the plane or dihedral angles is not so simple.

Multiplying (7) by $\sin\phi \sin\psi$, squaring and transposing, we have

$$2\Pi\cos\alpha \,.\, \sin^2\phi \sin^2\psi = 2\sin^2\phi \sin^2\psi - \sin^2\alpha \cos^2\theta \sin^2\phi \sin^2\psi - \sin^2\beta \sin^2\phi \sin^2\psi - \sin^2\gamma \sin^2\phi \sin^2\psi,$$

or

$$2\cos\alpha \sin\phi \sin\psi \sqrt{(\sin^2\phi - \sin^2\alpha \sin^2\theta)(\sin^2\psi - \sin^2\alpha \sin^2\theta)}$$
$$= 2\sin^2\phi \sin^2\psi - \sin^2\alpha \cos^2\theta \sin^2\phi \sin^2\psi - \sin^2\alpha \sin^2\theta(\sin^2\phi + \sin^2\psi),$$

which may be reduced to

$$4\sin^4\alpha \sin^2\theta - \sin^2\alpha \left\{ 2\Sigma \sin^2\theta - \left(1 + \Sigma \frac{\cos 2\theta}{\sin^4\theta}\right) \Pi \sin^2\theta \right\} + 4\sin^2\phi \sin^2\psi = 0 \ldots(9)$$

The determination of A in terms of θ, ϕ, ψ results in the same equation; consequently the two roots of this equation give both $\sin^2\alpha$ and $\sin^2 A$. This might have been inferred from the symmetry of the relations (7) and (8).

The different signs in the product of the three cosines of α, β, γ and of A, B, C in (7) and (8) show that when the three perpendiculars from the angular points of a spherical triangle on the opposite sides are given, there are two such triangles, each of which is the polar of the other.

6. The following are the most important expressions for the volume of the tetrahedron:

$$6V = 2p_1\Delta_1 = abc \sin\theta_1 \sin\alpha_1,$$

which from (7) $= abc\sqrt{1 - \cos^2\alpha_1 - \cos^2\beta_2 - \cos^2\gamma_3 + 2\cos\alpha_1 \cos\beta_2 \cos\gamma_3}$ (10)

$$= abc \sin\beta_2 \sin\gamma_3 \sin A, \text{ from (3)}, \ldots\ldots(11)$$

$$= 4\Delta_2\Delta_3 \sin A/a = 4\Delta_0\Delta_1 \sin X/x \text{ (by symmetry)}$$

$$= 4\Delta_1\Delta_3 \sin B/b = 4\Delta_0\Delta_2 \sin Y/y = \text{etc.} \ldots\ldots(12)$$

Hence we obtain

$$\frac{ax}{\sin A \sin X} = \frac{by}{\sin B \sin Y} = \frac{cz}{\sin C \sin Z} = \frac{4\Delta_0\Delta_1\Delta_2\Delta_3}{9V^2} = h^2. \ldots\ldots(13)$$

From (10) we have

$$36V^2 = a^2b^2c^2 \begin{vmatrix} 1, & \cos\gamma_3, & \cos\beta_2 \\ \cos\gamma_3, & 1, & \cos\alpha_1 \\ \cos\beta_2, & \cos\alpha_1, & 1 \end{vmatrix};$$

$$\therefore \quad 288V^2 = \begin{vmatrix} 2a^2, & a^2+b^2-z^2, & c^2+a^2-y^2 \\ a^2+b^2-z^2, & 2b^2, & b^2+c^2-x^2 \\ c^2+a^2-y^2, & b^2+c^2-x^2, & 2c^2 \end{vmatrix}, \ldots\ldots(14)$$

which, if expanded, gives

$$144V^2 = \Sigma a^2x^2(b^2+y^2+c^2+z^2-a^2-x^2) - \Sigma x^2y^2z^2. \ldots\ldots(15)$$

See Salmon's *Geometry of Three Dimensions*, Art. 53.

In the *Educational Times* for May, 1897, part of problem 13494 by Wolstenholme is

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ b^2+c^2-x^2, & y^2+z^2-x^2, & a^2-b^2-z^2, & a^2-y^2-c^2 \\ c^2+a^2-y^2, & b^2-a^2-z^2, & z^2+x^2-y^2, & b^2-c^2-x^2 \\ a^2+b^2-z^2, & c^2-a^2-y^2, & c^2-x^2-b^2, & x^2+y^2-z^2 \end{vmatrix} = -1152V^2,$$

which agrees with (14).

7. Besides the four identities (1) the twelve plane angles must be connected by three equations, which may be thus found:

$$x/\sin\alpha_0 = y/\sin\beta_0 = z/\sin\gamma_0 \text{ (from triangle } ABC),$$
$$x/\sin\alpha_1 = b/\sin\beta_1 = c/\sin\gamma_1 \text{ (from triangle } OBC),$$
$$a/\sin\alpha_2 = y/\sin\beta_2 = c/\sin\gamma_2 \text{ (from triangle } OCA),$$
$$a/\sin\alpha_3 = b/\sin\beta_3 = z/\sin\gamma_3 \text{ (from triangle } OAB);$$
$$\therefore\ a/\sin\alpha_3 = x\sin\beta_1/\sin\alpha_1\sin\beta_3 = x\sin\gamma_0/\sin\alpha_0\sin\gamma_3,$$

and
$$a/\sin\alpha_2 = x\sin\beta_0/\sin\alpha_0\sin\beta_2 = x\sin\gamma_1/\sin\alpha_1\sin\gamma_2;$$

$$\therefore\ \frac{\sin\alpha_2\sin\beta_0}{\sin\alpha_0\sin\beta_2} = \frac{\sin\alpha_3\sin\beta_1}{\sin\alpha_1\sin\beta_3} = \frac{\sin\alpha_3\sin\gamma_0}{\sin\alpha_0\sin\gamma_3} = \frac{\sin\alpha_2\sin\gamma_1}{\sin\alpha_1\sin\gamma_2} = \frac{a}{x} \quad \ldots\ldots\ldots(16)$$

And of course there are similar expressions equal to b/y and c/z. If however five plane angles are given the rest cannot be found *directly* from these relations.

These relations (16) are equivalent to the statement that: Considering any three triangles having a common vertex, the product of the ratios of the sines of the two angles not at the common vertex in each triangle taken in order is equal to unity.

It should be noticed that from the above equations the ratio of the lengths of any pair of edges can be expressed in terms of the sines of some of the plane angles.

8. Relations between the θ, ϕ, ψ angles can be found by expressing the lengths of the perpendiculars from the vertices on the opposite faces, thus:

$$p_0 = a\sin\theta_0 = b\sin\phi_0 = c\sin\psi_0,$$
$$p_1 = a\sin\theta_1 = y\sin\phi_1 = z\sin\psi_1,$$
$$p_2 = x\sin\theta_2 = b\sin\phi_2 = z\sin\psi_2,$$
$$p_3 = x\sin\theta_3 = y\sin\phi_3 = c\sin\psi_3.$$

From which we may obtain, as in case of the plane angles, that

$$\frac{a}{x} = \frac{\sin\theta_2\sin\phi_0}{\sin\theta_0\sin\phi_2} = \frac{\sin\theta_3\sin\phi_1}{\sin\theta_1\sin\phi_3} = \frac{\sin\theta_3\sin\psi_0}{\sin\theta_0\sin\psi_3} = \frac{\sin\theta_2\sin\psi_1}{\sin\theta_1\sin\psi_2} \quad \ldots\ldots\ldots(17)$$

These relations are identical in form to those just found (16) existing between the plane angles, and may be expressed in the statement that: Considering any three edges in the same plane, the product of the ratios of the sines of the angles which each edge makes with the two opposite faces taken in order is equal to unity.

As in the case of the plane angles, the above equations enable us to express the ratio of the lengths of any pair of edges in terms of the sines of the θ, ϕ, ψ angles.

The above give only three independent relations between these twelve angles; the remaining four may be found from equations corresponding to (9). As there mentioned, (9) is a quadratic in $\sin^2 A$ as well as in $\sin^2\alpha$; and

if we form the corresponding quadratic in $\sin^2 A$ from the angles at the vertex A, the eliminant may be written down, but it is extremely cumbrous, and does not look as if it would reduce to anything reasonable. Of course six such equations could be found of which two must result from the other four by the aid of (17).

9. The relation between the six dihedral angles may be found by projecting in turn three of the faces on the fourth. Hence we obtain

$$\left.\begin{aligned} -\Delta_0 \quad &+\Delta_1 \cos X+\Delta_2 \cos Y+\Delta_3 \cos Z=0 \\ \Delta_0 \cos X-\Delta_1 \quad &+\Delta_2 \cos C+\Delta_3 \cos B=0 \\ \Delta_0 \cos Y+\Delta_1 \cos C-\Delta_2 \quad &+\Delta_3 \cos A=0 \\ \Delta_0 \cos Z+\Delta_1 \cos B+\Delta_2 \cos A-\Delta_3 \quad &=0 \end{aligned}\right\} \quad \text{.....(A)}$$

and the eliminant is

$$\begin{vmatrix} -1, & \cos X, & \cos Y, & \cos Z \\ \cos X, & -1, & \cos C, & \cos B \\ \cos Y, & \cos C, & -1, & \cos A \\ \cos Z, & \cos B, & \cos A, & -1 \end{vmatrix} = 0, \quad \text{.....(18)}$$

which expands into

$$1-\Sigma \cos^2 A-2\Sigma \cos A \cos B \cos C$$
$$-2\Sigma \cos B \cos Y \cos C \cos Z+\Sigma \cos^2 A \cos^2 X=0.$$

If five of these angles are given, this is a quadratic for determining the sixth. If this equation be solved for $\cos A$, we have

$$\sin^2 X \cos A=-(\cos B \cos C+\cos Y \cos Z+\cos B \cos X \cos Y+\cos C \cos Z \cos X)$$
$$\pm\sqrt{(1-\cos^2 B-\cos^2 Z-\cos^2 X-2\cos B \cos Z \cos X)}$$
$$\times\sqrt{(1-\cos^2 C-\cos^2 X-\cos^2 Y-2\cos C \cos X \cos Y)}. \quad \text{.....(19)}$$

This result might also be obtained from (5) and (6); and from these equations we can see that only one root in (19) is admissible: for from (5) at the vertex B, $\cos \gamma_1$, and consequently γ_1, has but one value; similarly at the vertex C, β_1 has but one value; therefore from (1) α_1 has but one value; and therefore from the first equation of (5) A has only one value. It thus follows that if five *specified* dihedral angles are given and one edge, only one tetrahedron is determined.

10. To find the ratios of the lengths of the edges in terms of the dihedral angles.

Denoting by $\delta(\cos A)$, $\delta(\cos X)$, etc., the co-factors of $\cos A$, $\cos X$, etc., in the development of the determinant (18), we obtain from equations (A)

$$\begin{aligned} \Delta_1/\delta(\cos X)&=\Delta_2/\delta(\cos Y)=\Delta_3/\delta(\cos Z), \\ \Delta_0/\delta(\cos X)&=\Delta_2/\delta(\cos C)=\Delta_3/\delta(\cos B), \\ \Delta_0/\delta(\cos Y)&=\Delta_1/\delta(\cos C)=\Delta_3/\delta(\cos A), \\ \Delta_0/\delta(\cos Z)&=\Delta_1/\delta(\cos B)=\Delta_2/\delta(\cos A). \end{aligned}$$

Noting that $\delta(\cos A).\delta(\cos X)=\delta(\cos B).\delta(\cos Y)=\delta(\cos C).\delta(\cos Z)$, and that the co-factors are all negative quantities, we obtain

$$\Delta_0\Delta_1/\delta(\cos X)=\Delta_2\Delta_3/\delta(\cos A)=-\{\Delta_0\Delta_1\Delta_2\Delta_3/\delta(\cos A).\delta(\cos X)\}^{\frac{1}{2}}$$
$$=\Delta_0\Delta_2/\delta(\cos Y)=\Delta_0\Delta_3/\delta(\cos Z)=\Delta_1\Delta_2/\delta(\cos C)=\Delta_1\Delta_3/\delta(\cos B).$$

Combining these equations with (12) we have

$$*a/\sin A\delta(\cos A)=b/\sin B\delta(\cos B)=\ldots=-h/\{\delta(\cos A)\delta(\cos X)\}^{\frac{1}{2}} \quad \text{.....(20)}$$

* For this result I have to thank Dr. F. S. Macaulay.

11. The areas of the faces may also be expressed in terms of the dihedral angles and the quantity h.

From (6) we have

$$\begin{aligned}(1-\cos^2A-\cos^2B-\cos^2C-2\cos A\cos B\cos C)^3&=(\sin a_1\sin\beta_2\sin\gamma_3\sin^2A\sin^2B\sin^2C)^2\\&=\left\{\frac{8\Delta_1\Delta_2\Delta_3}{a^2b^2c^2}\times\left(\frac{3Va}{2\Delta_2\Delta_3}\cdot\frac{3Vb}{2\Delta_3\Delta_1}\cdot\frac{3Vc}{2\Delta_1\Delta_2}\right)^2\right\}^2,\text{ from (12),}\\&=(8\Delta_0{}^3/h^6)^2;\end{aligned}$$

$$\therefore\ *1-\cos^2A-\cos^2B-\cos^2C-2\cos A\cos B\cos C=4\Delta_0{}^2/h^4;$$

$$\therefore\ \frac{\Delta_0{}^2}{\begin{vmatrix}-1, & \cos B, & \cos C\\ \cos B, & -1, & \cos A\\ \cos C, & \cos A, & -1\end{vmatrix}}=\frac{\Delta_1{}^2}{\begin{vmatrix}-1, & \cos Y, & \cos Z\\ \cos Y, & -1, & \cos A\\ \cos Z, & \cos A, & -1\end{vmatrix}}$$

$$=\frac{\Delta_2{}^2}{\begin{vmatrix}-1, & \cos X, & \cos Z\\ \cos X, & -1, & \cos B\\ \cos Z, & \cos B, & -1\end{vmatrix}}=\frac{\Delta_3{}^2}{\begin{vmatrix}-1, & \cos X, & \cos Y\\ \cos X, & -1, & \cos C\\ \cos Y, & \cos C, & -1\end{vmatrix}}=-\frac{h^4}{4}\quad\ldots\ldots(21)$$

12. The following is due to Wolstenholme. Shortly before his death I happened to meet him, and he wrote down the result as a problem for me to work at. I sent it to the *Educational Times*, where it appeared June, 1897, No. 13521:

$$\frac{(a+x)^2+u}{\sin^2\tfrac12(A+X)}=\frac{(a-x)^2+u}{\sin^2\tfrac12(A-X)}=\frac{(b+y)^2+u}{\sin^2\tfrac12(B+Y)}=\ldots=4h^2,\ \ldots\ldots\ldots\ldots(22)$$

where

$$u\equiv(\Delta_0+\Delta_1+\Delta_2+\Delta_3)(\Delta_0+\Delta_1-\Delta_2-\Delta_3)(\Delta_0+\Delta_2-\Delta_1-\Delta_3)(\Delta_0+\Delta_3-\Delta_1-\Delta_2)/9V^2,$$

$$\begin{aligned}9V^2u&=\{(\Delta_0+\Delta_1)^2-(\Delta_2+\Delta_3)^2\}\{(\Delta_0-\Delta_1)^2-(\Delta_2-\Delta_3)^2\}\\&=(\Delta_0{}^2+\Delta_1{}^2-\Delta_2{}^2-\Delta_3{}^2)^2-4(\Delta_0\Delta_1-\Delta_2\Delta_3)^2.\end{aligned}$$

Also by multiplying equations (A) [§ 9] by Δ_0, Δ_1, $-\Delta_2$, $-\Delta_3$ in order, and adding we obtain

$$\Delta_0{}^2+\Delta_1{}^2-\Delta_2{}^2-\Delta_3{}^2=2\Delta_0\Delta_1\cos X-2\Delta_2\Delta_3\cos A.\ldots\ldots\ldots\ldots\ldots(23)$$

Hence
$$\begin{aligned}9V^2u&=4(\Delta_0\Delta_1\cos X-\Delta_2\Delta_3\cos A)^2-4(\Delta_0\Delta_1-\Delta_2\Delta_3)^2\\&=8\Delta_0\Delta_1\Delta_2\Delta_3(1-\cos A\cos X)-4(\Delta_0{}^2\Delta_1{}^2\sin^2X+\Delta_2{}^2\Delta_3{}^2\sin^2A)\\&=8\Delta_0\Delta_1\Delta_2\Delta_3(1-\cos A\cos X)-9(a^2+x^2)V^2,\text{ from (12);}\\\therefore\ (a+x)^2+u&=2h^2(1-\cos A\cos X)+2ax\\&=2h^2\{1-\cos(A+X)\},\text{ from (13),}\\&=4h^2\sin^2\tfrac12(A+X).\end{aligned}$$

Similarly $\qquad(a-x)^2+u=4h^2\sin^2\tfrac12(A-X).$

And so for $B\pm Y$, $C\pm Z$.

This investigation has been abbreviated from my own solution by a hint received from the solution of H. W. Curjel, M.A., in the reprint of the *Mathematical Times Solutions*, vol. LXVIII., p. 102.

13. The *Educational Times*, September, 1897.

13605. (The late Professor Wolstenholme, M.A., Sc.D.)—"*Tetrahedron, of course.* The equations I have been looking for for years:

$$\begin{aligned}[(b\pm y)^2-(c\pm z)^2]\sin^2\tfrac12(A\pm X)+[(c\pm z)^2-(a\pm x)^2]\sin^2\tfrac12(B\pm Y)\\+[(a\pm x)^2-(b\pm y)^2]\sin^2\tfrac12(C\pm Z)\equiv0;\ldots(24)\end{aligned}$$

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* For this result I have to thank Dr. F. S. Macaulay.

the ambiguities being always of the same sign in any term $(a \pm x)^2$ as in the corresponding term $\sin^2 \frac{1}{2}(A \pm X)$. The relations between the lengths of edges and dihedral angles are

$$\frac{(a+x)^2}{k+\sin^2\frac{1}{2}(A+X)}=\frac{(b+y)^2}{k+\sin^2\frac{1}{2}(B+Y)}=\frac{(c+z)^2}{k+\sin^2\frac{1}{2}(C+Z)}$$
$$=\frac{(a-x)^2}{k+\sin^2\frac{1}{2}(A-X)}=\frac{(b-y)^2}{k+\sin^2\frac{1}{2}(B-Y)}=\frac{(c-z)^2}{k+\sin^2\frac{1}{2}(C-Z)}=4h^2.$$

. . . . Can any one supply the exact value of k? I cannot at present."

["I think I have now accomplished all I can in the theory of the tetrahedron, and propose to pretermit the study for some time. There is, however, still a great deal to be done in the goniometry of the tetrahedron, and I expect sometime the theory of elliptic functions will start from it. I am too old and broken to have any hope of accomplishing this myself, or I would not divulge the idea." J. W., Oct. 22, 1889.]

Starting with (22), namely,

$$\frac{(a+x)^2+u}{\sin^2\frac{1}{2}(A+X)}=\frac{(b+y)^2+u}{\sin^2\frac{1}{2}(B+Y)}=\frac{(c+z)^2+u}{\sin^2\frac{1}{2}(C+X)};$$

it at once follows that each of these fractions

$$=\frac{0}{[(b+y)^2-(c+z)^2]\sin^2\frac{1}{2}(A+X)+[(c+z)^2-(a+x)^2]}$$
$$\times\sin^2\tfrac{1}{2}(B+Y)+[(a+x)^2-(b+y)^2]\sin^2\tfrac{1}{2}(C+Z),$$

which establishes the first identity when the upper signs are taken; and similarly when the lower signs are taken.

Again, since $$\frac{(a+x)^2+u}{\sin^2\frac{1}{2}(A+X)}=4h^2;$$

$$\therefore\ (a+x)^2=4h^2\sin^2\tfrac{1}{2}(A+X)-u.$$

If we let $u=-4h^2k$ we obtain $\dfrac{(a+x)^2}{k+\sin^2\frac{1}{2}(A+X)}=4h^2$, where

$$k=-\frac{(\Delta_0+\Delta_1+\Delta_2+\Delta_3)(\Delta_0+\Delta_1-\Delta_2-\Delta_3)(\Delta_0+\Delta_2-\Delta_1-\Delta_3)(\Delta_0+\Delta_3-\Delta_1-\Delta_2)}{16\Delta_0\Delta_1\Delta_2\Delta_3}.$$

And similarly for the equality of the other fractions.

It is evident that when Wolstenholme set me (22) he had found the value of k.

14. The following is also by Wolstenholme. See *Educational Times*, July, 1897, No. 13551:

"$$\frac{\sin(B+Y)+\sin(C+Z)}{b+y+c+z}$$
$$=\frac{3V}{32\Delta_0\Delta_1\Delta_2\Delta_3}[\{a^2+(y-c)(b-z)\}\{x^2+(b-c)(y-z)\}-(by+cz)(b+y-c-z)^2],\quad(25)$$

and any equation derived from this by cyclical permutation or by changing throughout the signs of any dihedral angles and the corresponding (opposite) edges is also true."

From (2) $\sin\gamma_3\sin a_1\cos B=\cos\beta_1-\cos\gamma_3\cos a_1$;

$$\therefore\ 16\Delta_1\Delta_3\cos B=2b^2(c^2+a^2-y^2)-(a^2+b^2-z^2)(b^2+c^2-x^2)$$
$$=b^2(c^2+z^2+a^2+x^2-b^2-y^2)+a^2x^2-b^2y^2+c^2z^2-c^2a^2-z^2x^2.$$

Similarly,

$$16\Delta_0\Delta_2 \cos Y = y^2(c^2+z^2+a^2+x^2-b^2-y^2)+a^2x^2-b^2y^2+c^2z^2-a^2z^2-c^2x^2,$$

$$16\Delta_1\Delta_2 \cos C = c^2(a^2+x^2+b^2+y^2-c^2-z^2)+a^2x^2+b^2y^2-c^2z^2-a^2b^2-x^2y^2,$$

$$16\Delta_0\Delta_3 \cos Z = z^2(a^2+x^2+b^2+y^2-c^2-z^2)+a^2x^2+b^2y^2-c^2z^2-b^2x^2-a^2y^2. \quad (26)$$

And from (12),

$$3V = 2\Delta_1\Delta_3 \sin B/b = 2\Delta_0\Delta_2 \sin Y/y = 2\Delta_1\Delta_2 \sin C/c = 2\Delta_0\Delta_3 \sin Z/z.$$

Hence $32\Delta_0\Delta_1\Delta_2\Delta_3\{\sin(B+Y)+\sin(C+Z)\}/3V$

$$\begin{aligned}
&= by(b+y)(c^2+z^2+a^2+x^2-b^2-y^2)+a^2x^2(b+y)\\
&\quad -b^2y^2(b+y)+c^2z^2(b+y)-a^2(bz^2+c^2y)-x^2(bc^2+yz^2)\\
&\quad +cz(c+z)(a^2+x^2+b^2+y^2-c^2-z^2)+a^2x^2(c+z)\\
&\quad +b^2y^2(c+z)-c^2z^2(c+z)-a^2(cy^2+b^2z)-x^2(b^2c+y^2z),
\end{aligned}$$

and on reduction

$$=(b+y+c+z)[\{a^2+(y-c)(b-z)\}\{x^2+(b-c)(y-z)\}-(b+y-c-z)^2(by+cz)].$$

And of course any equation formed by cyclical permutation is also true; but in changes of sign, it must be observed that it is not by simultaneous changes of B and y, but of B and b, that the resulting equation still holds; so that instead of 'opposite' in Wolstenholme's enunciation we should read 'adjacent.'

15. Let ABC be what we have called the spherical triangle at O, and suppose a line through O parallel to the side BC of the tetrahedron meet the sphere in D. OB, OC, and OD are in the same plane, and consequently D lies on the side BC of the spherical triangle ABC, and $CD=\beta_1$. AD is either λ or its supplement. Suppose $AD=\pi-\lambda$, then from the triangle ABC

$$\cos\gamma_3 = \cos\alpha_1\cos\beta_2+\sin\alpha_1\sin\beta_2\cos C,$$

and from ACD $\quad -\cos\lambda = \cos\beta_2\cos\beta_1-\sin\beta_2\sin\beta_1\cos C;$

$$\therefore \cos\gamma_3\sin\beta_1-\cos\lambda\sin\alpha_1 = \cos\beta_2(\sin\beta_1\cos\alpha_1+\sin\alpha_1\cos\beta_1)$$

$$= \cos\beta_2\sin\gamma_1, \text{ from (1);}$$

$$\therefore \cos\lambda = \cos\gamma_3 \cdot \sin\beta_1/\sin\alpha_1 - \cos\beta_2 \cdot \sin\gamma_1/\sin\alpha_1 \quad \ldots\ldots\ldots(27)$$

$$= \frac{a^2+b^2-z^2}{2ab}\cdot\frac{b}{x}-\frac{c^2+a^2-y^2}{2ca}\cdot\frac{c}{x}$$

$$= \frac{b^2+y^2-c^2-z^2}{2ax}. \quad \ldots\ldots\ldots(28)$$

By cyclic changes we obtain

$$\cos\mu = \cos\alpha_1\sin\gamma_2/\sin\beta_2-\cos\gamma_3\sin\alpha_2/\sin\beta_2 = (c^2+z^2-a^2-x^2)/2by, \quad \ldots(29)$$

$$\cos\nu = \cos\beta_2\sin\alpha_3/\sin\gamma_3-\cos\alpha_1\sin\beta_3/\sin\gamma_3 = (a^2+x^2-b^2-y^2)/2cz. \quad \ldots(30)$$

Hence $$\Sigma ax\cos\lambda = 0. \quad \ldots\ldots\ldots(31)$$

With the sign we have adopted for $\cos\lambda$, the other values are

$$\begin{aligned}
\cos\lambda &= \cos\gamma_2\sin\beta_0/\sin\alpha_0-\cos\beta_3\sin\gamma_0/\sin\alpha_0\\
&= \cos\gamma_1\sin\beta_3/\sin\alpha_3-\cos\beta_0\sin\gamma_3/\sin\alpha_3\\
&= \cos\gamma_0\sin\beta_2/\sin\alpha_2-\cos\beta_1\sin\gamma_2/\sin\alpha_2.
\end{aligned}$$

And similarly for $\cos\mu$ and $\cos\nu$.

16. To find the shortest distances (the perpendiculars l, m, n) between the three pairs of opposite edges.

Through O draw OD parallel and equal to CB: join AD, BD. Then $OCBD$ is a parallelogram, and consequently the tetrahedra $AOBD$, $AOBC$ are equal. Also the perpendicular from B on AOD is l, and the area of AOD is $\frac{1}{2}ax\sin\lambda$;

$$\therefore \quad lax\sin\lambda = 6V;$$

$$\therefore \quad l = 6V/ax\sin\lambda, \quad m = 6V/by\sin\mu, \quad n = 6V/cz\sin\nu \ldots\ldots\ldots\ldots (32)$$

17. To find the distances between the mid-points of the opposite edges.

The mid-points of the six edges taken four by four are obviously the angular points of three parallelograms of which d_1, d_2, d_3 are, two by two, the diagonals, which are therefore concurrent and bisect each other.

Also d_1, $b/2$, $y/2$ are sides of a triangle, the angle opposite d_1 being μ or $\pi-\mu$. Hence

$$4d_1^2 = b^2+y^2 \pm 2by\cos\mu = b^2+y^2 \pm (c^2+z^2-a^2-x^2), \text{ from (29).}$$

Similarly $4d_1^2 = c^2+z^2 \mp 2cz\cos\nu = c^2+z^2 \mp (a^2+x^2-b^2-y^2)$, from (30).

As these results must be identical, it is obvious that the upper signs must be taken throughout, so that

$$4d_1^2 = b^2+y^2+c^2+z^2-a^2-x^2. \ldots\ldots\ldots\ldots (33)$$

Similarly for $4d_2^2$ and $4d_3^2$;

$$\therefore \quad 4(d_3^2-d_2^2) = 2(b^2+y^2-c^2-z^2); \ldots\ldots\ldots\ldots (34)$$

$$\therefore \quad d_3^2-d_2^2 = ax\cos\lambda, \text{ from (28).}$$

Also $$2(d_3^2+d_2^2) = a^2+x^2. \ldots\ldots\ldots\ldots (35)$$

And similarly for the other pairs.

18. To find the radius R of the circum-sphere.

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the circum-centre referred to OA, OB, OC as coordinate axes; then we have (neglecting the subscripts of α_1, β_2, γ_3)

$$R^2 = \Sigma\bar{x}^2 + 2\Sigma\bar{y}\bar{z}\cos a.$$

Also $R^2 = (\bar{x}-a)^2+\bar{y}^2+\bar{z}^2+2\bar{y}\bar{z}\cos a+2\bar{z}(\bar{x}-a)\cos\beta+2(\bar{x}-a)\bar{y}\cos\gamma.$

Subtracting we get $$\bar{x}+\bar{y}\cos\gamma+\bar{z}\cos\beta = \tfrac{1}{2}a.$$

Similarly $$\bar{x}\cos\gamma+\bar{y}+\bar{z}\cos a = \tfrac{1}{2}b,$$

$$\bar{x}\cos\beta+\bar{y}\cos a+\bar{z} = \tfrac{1}{2}c.$$

$$\therefore \quad \tfrac{1}{2}(a\bar{x}+b\bar{y}+c\bar{z}) = \bar{x}^2+\bar{y}^2+\bar{z}^2+2\bar{y}\bar{z}\cos a+2\bar{z}\bar{x}\cos\beta+2\bar{x}\bar{y}\cos\gamma = R^2.$$

Solving these three equations we have

$$2\bar{x}\begin{vmatrix} 1, & \cos\beta, & \cos\gamma \\ \cos\beta, & 1, & \cos a \\ \cos\gamma, & \cos a, & 1 \end{vmatrix} = a-b\cos\gamma-c\cos\beta-\cos a(a\cos a-b\cos\beta-c\cos\gamma);$$

$$\therefore \quad \frac{72V^2\bar{x}}{a^2b^2c^2} = a-b\frac{a^2+b^2-z^2}{2ab}-c\frac{c^2+a^2-y^2}{2ca}-\frac{b^2+c^2-x^2}{2bc}$$

$$\times\left(a\frac{b^2+c^2-x^2}{2bc}-b\frac{c^2+a^2-y^2}{2ca}-c\frac{a^2+b^2-z^2}{2ab}\right),$$

$$\text{or } 288V^2\bar{x}/a = 2b^2c^2(y^2+z^2-b^2-c^2)-(b^2+c^2-x^2)(b^2y^2+c^2z^2-a^2x^2-2b^2c^2)$$

$$= (c^2a^2+a^2b^2-2b^2c^2)x^2+(b^2c^2-b^4)y^2+(b^2c^2-c^4)z^2+x^2(b^2y^2+c^2z^2-a^2x^2).$$

Similarly $288V^2\bar{y}/b$ and $288V^2\bar{z}/c$ may be written down.

Hence $$288V^2(a\bar{x}+b\bar{y}+c\bar{z}) = 2\Sigma b^2c^2y^2z^2-\Sigma a^4x^4,$$

or $$576V^2R^2 = \Sigma ax\,\Pi(by+cz-ax). \ldots\ldots\ldots\ldots (36)$$

See Salmon's *Geometry of Three Dimensions*, Art. 55; also Frost and Wolstenholme's *Solid Geometry*, Art. 437.

19. It is obvious that $r=3V/(\Delta_0+\Delta_1+\Delta_2+\Delta_3)$, ...(37)

and $r_0=3V/(\Delta_1+\Delta_2+\Delta_3-\Delta_0)$, ...(38)

with similar expressions for r_1, r_2, r_3.

Hence $1/r_0+1/r_1+1/r_2+1/r_3=2/r$. .....(39)

20. The following problems are suggested as applications of this paper:

(1) All tetrahedra which have a common edge and whose opposite vertices are on two given straight lines parallel to this edge are equal in volume.

(2) If a line is equally inclined to OA, OB, OC, then this inclination is $\cot^{-1}(\cos\frac{1}{2}\beta_2\cos\frac{1}{2}\gamma_3\sin A/\sin\frac{1}{2}\alpha_1)$; and the conditions that the four lines at the four vertices should be concurrent are $a+x=b+y=c+z$.

(3) The conditions that a sphere should meet the four faces in their nine-point circles are that the opposite edges should be at right angles.

(4) $\Delta_0^2=\Sigma\Delta_1^2-2\Sigma\Delta_2\Delta_3\cos A$.

(Wolstenholme's *Mathematical Problems*, Second edition, No. 2021.)

(5) If $b=y$ and $c=z$, then $B=Y$ and $C=Z$. Is the converse true?

(6) If $\Delta_0+\Delta_1=\Delta_2+\Delta_3$, then

$$(a-x)/\sin\tfrac{1}{2}(X-A)=(b-y)/\sin\tfrac{1}{2}(Y-B)=(c-z)/\sin\tfrac{1}{2}(Z-C)=2h.$$

(*Educational Times*, October, 1900, 14675, by Wolstenholme.)

(7) $\cos\beta_0\cos C\sin Z-\cos\beta_1\sin C\cos Z=\cos\gamma_0\cos B\sin Y-\cos\gamma_1\sin B\cos Y$.

(8) $24V(a\cot A-x\cot X)=4d_1^2(a^2-x^2)-(b^2-y^2)(c^2-z^2)$.

(9) $\Sigma\cot\lambda/l=0$.

(10) The sines of the angles between the three planes, each through four mid-points of the edges, are $d_1mn/3V$, $d_2nl/3V$, $d_3lm/3V$.

(11) The volume of the tetrahedron described on any face of $OABC$, its other edges being d_1, d_2, d_3 (d_1 being opposite to a or x, etc.), is $\frac{1}{3}V$.

(12) The diameter of the circum-sphere in terms of the angles and edges at the vertex O, neglecting subscripts, is

$$\left(\frac{\Sigma a^2\sin^2\alpha-2\Sigma bc\sin\beta\sin\gamma\cos A}{1-\Sigma\cos^2\alpha+2\Pi\cos\alpha}\right)^{\frac{1}{2}}.$$

(13) $(1/r-1/r_0)(1/r-1/r_1)(1/r-1/r_2)(1/r-1/r_3)=4h^2/9V^2$.

(14) $1/r_0r_1+1/r_2r_3=h^2/\Delta_0\Delta_1+h^2/\Delta_2\Delta_3$.

(15) The tetrahedral equation of the circum-sphere is

$$x^2\beta\gamma+y^2\gamma\alpha+z^2\alpha\beta+a^2\alpha o+b^2\beta o+c^2\gamma o=0,$$

where $o=0$, $\alpha=0$, $\beta=0$, $\gamma=0$ are the equations of the faces opposite to O, A, B, C respectively. (Frost and Wolstenholme's *Solid Geometry*, Art. 435; see also Salmon, Art. 229.)

(16) If the four perpendiculars from the vertices on the opposite faces are concurrent, then the opposite edges are at right angles;

$$a^2+x^2=b^2+y^2=c^2+z^2,$$

and $$\cos A\cos X=\cos B\cos Y=\cos C\cos Z.$$

The tetrahedral coordinates of the point of concurrence are proportional to

$$\Delta_0(\sec A\sec B\sec C)^{\frac{1}{2}},\quad \Delta_1(\sec Y\sec Z\sec A)^{\frac{1}{2}},\text{ etc.}*$$

G. RICHARDSON.

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* For the extension of the modern Geometry of a triangle to special tetrahedra, see a memoir by Professor M. J. Neuberg in *Mémoires Couronnés, L'Académie Royale de Belgique*, tome xxxvii., Bruxelles, Janvier, 1886.

MATHEMATICAL NOTES.

104. [K. 20. e ; 4.] *On the construction of a triangle, given mutual ratios of cosines of angles.*

In any triangle the cosines of the angles are bound together by the relation

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$$

Hence if we are given the mutual ratios $\cos A : \cos B : \cos C$, say $= d : e : f$,

$$2\frac{ef}{d^2}\cos^3 A + \cos^2 A\left(1 + \frac{e^2}{d^2} + \frac{f^2}{d^2}\right) = 1,$$

or

$$2ef\cos^3 A + \cos^2 A\,(d^2 + e^2 + f^2) = d^2,$$

a CUBIC in $\cos A$, showing that it is impossible to get a geometrical construction for the angles of a triangle having given the mutual ratios of their cosines.

Now if $AI : BI : CI$ be given in a triangle, we have

$$\sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2} = 1/AI : 1/BI : 1/CI$$
$$= \text{given ratios.}$$

The triangle formed by joining the points of contact of the inscribed circle with the sides has its angles $= \frac{1}{2}(B+C), \frac{1}{2}(C+A), \frac{1}{2}(A+B)$.

Hence we are given the *ratios* of the *cosines* of this triangle, and no *geometrical* construction can be found for determining the species of this triangle.

R. F. DAVIS.

105. [M¹. 8.] *The Equiangular Spiral.*

The equation $r = ae^{\theta\cot\alpha}$ to the equiangular spiral is usually obtained by integrating $\frac{1}{r}\cdot\frac{dr}{d\theta} = \cot\alpha$. But as e is the limit of $(1+x)^{\frac{1}{x}}$ when x is evanescent, the equation to the curve could be obtained thus by elementary algebra.

Let the spiral be the ultimate form of a polygon whose sides subtend equal infinitesimal angles γ at the origin.

Then

$$\frac{r_1}{a} = \frac{\sin(\alpha+\gamma)}{\sin\alpha} = \frac{r_2}{r_1} = \ldots = \frac{r}{r_{n-1}};$$

$$\therefore\ \frac{r}{a} = \left\{\frac{\sin(\alpha+\gamma)}{\sin\alpha}\right\}^n$$
$$= (1+\gamma\cot\alpha)^{\frac{\theta}{\gamma}} \text{ (where } \theta = n\gamma)$$
$$= \{(1+\beta)^{1/\beta}\}^{\theta\cot\alpha} \text{ (where } \beta = \gamma\cot\alpha)$$
$$= e^{\theta\cot\alpha} \text{ in the limit.}$$

W. GALLATLY.

106. [C. 1. a.] *On the Permutability of Independent Differentiations.*

It is not easy to give a proof of the formula

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x}$$

by the differential calculus only, without assuming more than seems fairly implied in the definition of a differential coefficient. I think it would be most convenient to deduce the formula from the rule for differentiating an integral with respect to a parameter, since this rule depends on principles whose truth in general is readily admitted.

We have $$f(x, y)-f(a, y)=\int_a^x f_u(u, y)du.$$

Differentiate this with respect to y. Thus

$$f_y(x, y)-f_y(a, y)=\int_a^x f_{uy}(u, y)du.$$

Differentiate this with respect to x. Then

$$f_{yx}(x, y)=f_{xy}(x, y). \quad \text{Q.E.D.}$$

A. C. DIXON.

REVIEWS AND NOTICES.

La Fonction Gamma; Théorie, Histoire, Bibliographie. Par Maurice Godefroy. Paris, Gauthier-Villars, 1901; Pp. 94.

The author defines $\Gamma(x)$ as the limit for $n=\infty$ of the product

$$n^x(n\,!)/x(x+1)\ldots(x+n),$$

which he denotes by $\Pi(x)$. This use of a symbol, used in a different sense by Gauss, is unfortunate; besides, it is often convenient to indicate the number n explicitly, and I shall use $\Pi(n, x)$ in this sense here (it is not quite the same as the sense attributed to the symbol by Gauss). Pringsheim, however (*Math. Ann.*, Bd. 31), prefers to make use of the definite integral for $\Gamma(x)$; in the preface, M. Godefroy explains his reasons for adopting a different course; although these seem sufficient, I regret that M. Godefroy did not include a proof of the connection between the two definitions.

I have found M. Godefroy's monograph most interesting and useful; those who have had to hunt for various results through numerous books on analysis, will appreciate the convenience of having them collected in one neat volume. But still, I think that the analysis might be made clearer in a few places.

For instance, Wallis's limit for $\frac{1}{2}\pi$ is frequently used, but yet M. Godefroy does not point out that this limit is only a special case of his definition; thus

$$\pi^{\frac{1}{2}}=\Gamma(\tfrac{1}{2})=\operatorname*{Lt}_{n=\infty}\Pi(n, \tfrac{1}{2})=\operatorname*{Lt}_{n=\infty}[n^{\frac{1}{2}}(n\,!)/\tfrac{1}{2}\,.\,\tfrac{3}{2}\ldots(n+\tfrac{1}{2})]$$

$$=\operatorname*{Lt}_{n=\infty}[n^{\frac{1}{2}}(2\,.\,4\,.\,6\ldots 2n)/1\,.\,3\ldots(2n+1)],$$

which is equivalent to Wallis's limit. But, in most cases, it simplifies the work to avoid this limit; thus, in proving (p. 30) Legendre's formula

$$\Gamma(2x)\,\Gamma(\tfrac{1}{2})=2^{2x-1}\,\Gamma(x)\Gamma(x+\tfrac{1}{2}),$$

we have at once, by multiplication,

$$\frac{\Pi(n, x)\,.\,\Pi(n, x+\frac{1}{2})}{\Pi(n, \frac{1}{2})\,.\,\Pi(2n, 2x)}=\frac{1}{2^{2x-1}}\frac{2n+1}{2n+x+1},$$

so that the result follows by passing to the limit with $n=\infty$. Almost the same point occurs in the next section in the proof of Gauss's extension of Legendre's formula. Again, in obtaining Gudermann's and Stirling's results (p. 44), it is proved that

$$\log\Gamma(x)=\lambda+(x-\tfrac{1}{2})\log x-x+\omega(x)$$

where λ is a constant (to be found) and $\omega(x)$ is a function for which $\omega(\infty)=0$. To find λ we have, from this and Legendre's theorem given above,

$$\lambda=\tfrac{1}{2}\log 2+\log[x^{\frac{1}{2}}\,\Gamma(\tfrac{1}{2})\,\Gamma(x)/\,\Gamma(x+\tfrac{1}{2})]+\omega(2x)-2\omega(x).$$

Now, by a theorem due to Weierstrass (p. 16), we have

$$\operatorname*{Lt}_{x=\infty}[x^p\,\Gamma(x)/\Gamma(x+p)]=1$$

if p is any (finite) number. Hence $\lambda=\frac{1}{2}\log(2\pi)$, independently of Wallis's limit.

In addition to the gamma-function, M. Godefroy treats of several allied functions, such as Binet's function (the $\omega(x)$ referred to above), $\Phi(x)=\Gamma'(x)/\Gamma(x)$ and $\Psi(x)=\Phi'(x)$. Incidentally we note that a curve is given to represent the gamma-function (on p. 65). We have next a chapter dealing with expansions in power-series; those given include $\log\Gamma(1+x)$, $\Gamma(1+x)$, $1/\Gamma(1+x)$, etc.

The book closes with a chapter on miscellaneous applications, in which a special case (p. 76) is given (without reference) of Weierstrass's theorem as to the convergence of $\sum_{n=0}^{\infty} u_n x^n$ for points *on* the circle of convergence (see § 5, Satz vii. of Weierstrass's memoir, *Ueber die Theorie der analytischen Facultäten*). The omission is the more remarkable inasmuch as frequent reference is made to other parts of the same memoir.

M. Godefroy gives no account of Barnes's recent work on the double gamma-function; but references are given to his papers. T. J. I'A. BROMWICH.

Applied Mechanics. By GAETANO LANZA (Pp. 928). New York: John Wiley & Sons; London, Chapman & Hall, 1900.

This work is mainly a compendium (for the use of engineers) of facts and experimental results relating to strength of materials; as such it can hardly be adequately criticised here. Its practical value is indicated by the fact that the present edition is the eighth (the first was published in 1885).

Some parts deal with theoretical matters; and in places, the theory seems not entirely satisfactory. For instance, the treatment of "centrifugal force" (p. 81) is obscure; the acceleration of a particle moving uniformly in a circle is proved to be v^2/r *towards* the centre, and then the "centrifugal force" is taken as mv^2/r *away* from the centre, without giving any reason for the change of direction. Professor Lanza uses the "weight of a body, as a rule, in preference to its mass"; no doubt this is preferred by engineers, but the use of weight in defining "moment of inertia" (§ 89) seems uncalled for, and would lead to confusion when comparing with other writers. The abbreviation $\int_0^x \int_0^x w dx^2$ strikes us as unfamiliar, and the saving in space is not very great.

A large number of numerical problems will be found worked out; but unfortunately Professor Lanza follows the too common plan of giving answers to more significant figures than can be obtained from the *data*. This is greatly to be deplored, as it increases the computer's labour, and adds a spurious appearance of accuracy; for instance (p. 22), *five*-figure tables are used, but the answers contain *eight* figures. Again, in the theory of continuous beams, certain approximate results are deduced in the ordinary way; for instance, on p. 774, the terms neglected are to those retained in the ratio $1 : 10^2$ (roughly), but, nevertheless, Prof. Lanza gives *nine* significant figures in the answer. Such procedure must be misleading to the student. T. J. I'A. BROMWICH.

A Treatise on Elementary Statics for the Use of Schools and Colleges. By W. J. DOBBS, M.A. London, A. & C. Black, 1901. (Pp. xi, 311.)

It seems a pity that Mr. Dobbs did not take the opportunity of putting a little more life into his subject. But what is attempted is for the most part well done; that is to say, propositions and problems involving co-planar forces, covering the usual range of a school-book, are well worked out, with clear explanations and good figures. Virtual work is introduced in its proper place, also the "funicular" polygon, which might have been developed a little further with advantage. Why cannot we adopt Clifford's hint that short words should sometimes be used instead of always long ones? "Link" is moreover a better as well as a shorter word than "funicular," suggesting the stiffness which is needed instead of suggesting flexibility, which has to be put aside with an effort.

The author suggests that the proof of the parallelogram of forces, and hence of results depending on it, such as the principle of moments, should be postponed till a second reading. It would perhaps be a better course, if the parallelogram of forces is thought too difficult for a beginner, to go back to the old plan of beginning with a lever, as in the days of Whewell. A uniform bar obviously balances about its middle point; divide the bar into any two parts, each of which balances about its middle point. Thus we get the equality of moments in a way in which, to the most inexperienced eye, it will appear axiomatic. The picture given by Mach (p. 15) for this purpose, with the whole bar and each part of it suitably pivotted, is very neat and convincing.

Mr. Dobbs has anyhow judged wisely in not basing the parallelogram of forces on a theory of kinetics. His version of Duchayla's proof, which is quite suitable

for his purpose, is one of the best things in the book, and appears to be new. Does anyone know anything about Duchayla? or is his very brief communication to the Société Philomathique in 1804 his sole title to fame? W. H. MACAULAY.

Theoretical Mechanics. An Elementary Text-Book. By L. M. HOSKINS. Stanford University, Cal., 1900. (Pp. ix, 436.)

The primary object of the author is to meet the needs of students of Engineering, and he has produced a good and useful book, which may be recommended to any student who is more interested in the subject than in the mathematical puzzles which can be constructed out of it. He begins at the beginning with elementary statics, and includes a brief treatment of rigid bodies in two dimensions, which is practically far enough for a book of this class to go. The book is throughout elementary in the sense of being easy. Differential and integral calculus are freely used wherever they are wanted, but no more knowledge of these subjects is demanded than any reader can easily acquire.

The author steers clear of graphical statics, probably because he expects this part of the subject to be supplied by special books. But the result is to convey a somewhat false view of how some questions should be dealt with.

It is worth noticing that, for the composition of two couples acting upon a rigid body in planes inclined to one another, the method used is to make all the forces equal, so that the arms represent the moments, and to place the couples so that two forces balance one another along the line of intersection of the planes. This, the neatest and most obvious of all the ways in which the parallelogram law can be exhibited, has no pretension to novelty, but has been strangely neglected by writers of text-books. Where is there a book in which it is to be found?

In his account of the laws of motion the author is curiously perverse about the base relative to which the motion is to be reckoned, proposes the earth as the base for ordinary practical problems without attention to the corrections needed, and goes so far as to suggest an arbitrary choice. In other respects the treatment of this part of the subject is fairly good; in particular it may be noticed that due attention is directed to the classification of forces. The gap between the definition of force by motion and the principle of transmissibility is bridged over by an appeal to experience, which is hardly sufficient to deal with the point in question, and it might have been well to refer in Chapter IV. to the subsequent treatment of this point (not very explicit) at the end of the book. This gap having somehow been crossed, it might have been well to put the parallelogram of forces on a basis independent of motion.

The treatment of the motion of a rigid body is well and neatly done, but the author has shown rather too much self restraint in not developing it a little further. W. H. MACAULAY.

A Primer of Astronomy. By SIR ROBERT BALL. Pp. viii, 183. 1s. 6d. 1900. (Cam. Univ. Press.)

The name of the author is sufficient guarantee that the subject matter of this book is presented in excellent style. We are not sure however, that Sir Robert is at his best when writing a primer. His experience has been that of a lecturer, and as a lecturer on scientific subjects he has no rival. For the beginner some of the letterpress is much too difficult, unless he has a master to fall back on for the necessary explanations. The same lack of acquaintance with the uninstructed and youthful mind is no doubt the cause of the unsatisfactory character of some of the diagrams, figs. 19 and 21 being extreme instances of what we regard as failures from the point of view of the beginner. On the other hand, the eleven plates are beautiful, and are alone worth the eighteenpence demanded for the book.

La Selenografia antica e Moderna, studio storico-scientifico. By Prof. BELLINO CARRARA. Pp. 116. 1900. (Fusi, Pavia.)

This handy little *brochure* forms an excellent summary of selenography from the days of the earliest astronomers to the present time. It forms an interesting introduction to the study of the moon, and should be of use to those who wish to make themselves acquainted with the work of Schröter, Mayer, Schmidt, Nasmyth, Neisen, and others, whose names are connected with researches into the

nature of lunar phenomena. To one section in Chapter XIII. we cannot refrain from calling attention, as somewhat out of place in a scientific document: ***o quello della piccolezza della Terra rispetto agli altri mondi, in relazione al mistero dell' Incarnazione del Figliuolo di Dio.***

SOLUTION.

390. [K. 13. a.] *If $A_1, \ldots, A_6$ be any six points ranged in this order on the circumference of a circle, and O any point in space,*

$$\frac{OA_1^2}{\Pi(A_1A_s)}+\frac{OA_3^2}{\Pi(A_3A_s)}+\frac{OA_5^2}{\Pi(A_5A_s)}=\frac{OA_2^2}{\Pi(A_2A_s)}+\frac{OA_4^2}{\Pi(A_4A_s)}+\frac{OA_6^2}{\Pi(A_6A_s)},$$

where $$\Pi(A_1A_s)=A_1A_2\,.\,A_1A_3\,.\,A_1A_4\,.\,A_1A_5\,.\,A_1A_6, \text{ etc.}$$

SIR ROBERT BALL.

Solution by T. J. I'A. BROMWICH.

Take as the plane $z=0$, the plane of the circle, and let $(x, y), \ldots, (x_6, y_6)$ be the coordinates of the six points on the circle, the origin being at the centre; also write

$$\zeta_r=x_r+iy_r \quad (r=1, 2, \ldots, 6).$$

Then if $$x_r=a\cos\theta_r, \quad y_r=a\sin\theta_r,$$

we have $$A_rA_s=2a\sin\tfrac{1}{2}(\theta_r-\theta_s)=\frac{a}{i}\frac{\zeta_r-\zeta_s}{(\zeta_r\zeta_s)^{\frac{1}{2}}},$$

where we shall suppose the sign of A_rA_s determined so as to satisfy this equation. Now consider

$$A_1A_2\,.\,A_1A_3\,.\,\ldots\,.\,A_1A_6=\left(\frac{a}{i}\right)^5\frac{1}{\zeta_1^{\,2}}\,[\zeta_1\zeta_2\ldots\zeta_6]^{-\frac{1}{2}}(\zeta_1-\zeta_2)(\zeta_1-\zeta_3)\ldots(\zeta_1-\zeta_6)$$

$$=\frac{1}{m_1}, \text{ say,}$$

and if $m_2, m_3, \ldots, m_6$ are defined similarly, we have

$$\Sigma m_rf(\zeta_r)=\Sigma\frac{i}{a^5}\,(\zeta_1\zeta_2\ldots\zeta_6)^{\frac{1}{2}}\frac{\zeta_r^{\,2}f(\zeta_r)}{(\zeta_r-\zeta_1)\ldots(\zeta_r-\zeta_6)},$$

where in the denominator the vanishing factor is of course omitted. Now this summation is equal to the coefficient of ζ^{-3} in the expansion, in powers of $(1/\zeta)$, of

$$\frac{i}{a^5}\,(\zeta_1\zeta_2\ldots\zeta_6)^{\frac{1}{2}}\Sigma\frac{1}{\zeta-\zeta_r}\frac{f(\zeta_r)}{(\zeta_r-\zeta_1)\ldots(\zeta_r-\zeta_6)},$$

or of $$\frac{i}{a^5}\,(\zeta_1\zeta_2\ldots\zeta_6)^{\frac{1}{2}}\frac{f(\zeta)}{(\zeta-\zeta_1)(\zeta-\zeta_2)\ldots(\zeta-\zeta_6)},$$

provided that $f(\zeta)$ is a polynomial in ζ of degree less than six. If now the highest term in $f(\zeta)$ is ζ^2, the coefficient of ζ^{-3} in the expansion will be zero and so

$$\Sigma m_rf(\zeta_r)=0,$$

or, since all the m's are real,

$$\Sigma m_r=0, \quad \Sigma m_rx_r=0, \quad \Sigma m_ry_r=0, \quad \Sigma m_r(x_r^2-y_r^2)=0, \quad \Sigma m_rx_ry_r=0.$$

It follows that

$$\Sigma m_r[(x-x_r)^2+(y-y_r)^2+z^2]=\Sigma m_r(x_r^2+y_r^2)=a^2\Sigma m_r=0.$$

In the same way the moment of inertia of six particles of masses m_1, m_2, ..., m_6 at these six points about any line is zero.

Solution by Proposer.

It is easy to shew that six parallel forces at $A_1 \ldots A_6$ will equilibrate if the forces be alternately positive and negative, and equal to $OA_1^2/\pi(A_1A_s)$ respectively. This is equivalent to the theorem as stated. [See Vol. II., page 26.]

The formula was suggested as follows:—It was proved by Mr. J. H. Grace that the six screws of a co-reciprocal system are parallel to generators of a quadric cone.

Let O be the vertex of the cone, and let $A_1 \ldots A_6$ be the intersections of these generators with any circular section. Then the pitches of the six co-reciprocal screws are represented by $\pi(A_1A_s)/OA_1^2$ and similar quantities alternately positive and negative. It is, however, known that the sum of the reciprocals of the pitches of such a system must be zero.

The statement of this is Question 390.

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GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO.